Theorem 3

$$D\frac{\partial f}{\partial x'} = \frac{dF}{dX}D.$$

Proof

We rewrite the result of lemma 2 as

$$\frac{dF}{dX}D\,dx = D\,df = D\frac{\partial f}{\partial x'}dx.$$

Omitting the arbitrary dx we obtain the result of theorem 3.

□

The basic result follows as a corollary, viz.

Corollary 4

$$\frac{dF}{dX} = D\frac{\partial f}{\partial x'}D^+.$$

Proof

Postmultiplication of the result of theorem 3 by D^+ yields, by virtue of lemma 1,

$$D\frac{\partial f}{\partial x'}D^+ = \frac{dF}{dX}DD^+ = \frac{1}{2}\frac{dF}{dX}(I+K) = \frac{dF}{dX}.$$

□

3. SOME ILLUSTRATIONS

(1) Kollo & von Rosen (2000) find

$$\frac{dF}{dX} = \frac{1}{2}\lambda(I+K), \quad \lambda \text{ a constant,}$$

for $F = \lambda X$. (Their result 2.9).

This can be derived succinctly in the following way. Differentiation of F yields $dF = \lambda dX$, which leads to $d\,\mathrm{vec}F = \lambda d\,\mathrm{vec}X$ and subsequently to $df = \lambda dx$. Hence $\frac{\partial f}{\partial x'} = \lambda I$ and $\frac{dF}{dX} = \lambda DD^+ = \frac{1}{2}\lambda(I+K)$ by corollary 4.

(2) $$\frac{dF}{dX} = -\tfrac{1}{2}(I+K)\left(X^{-1}\otimes X^{-1}\right) \text{ for } F = X^{-1}.$$

Proceeding as before we get

$$\begin{aligned} dF &= -X^{-1}(dX)X^{-1}, \\ d\,\mathrm{vec}F &= -\left(X^{-1}\otimes X^{-1}\right) d\,\mathrm{vec}X, \\ df &= -D^{+}\left(X^{-1}\otimes X^{-1}\right) D\,dx \quad \text{and} \\ \frac{\partial f}{\partial x'} &= -D^{+}\left(X^{-1}\otimes X^{-1}\right) D. \end{aligned}$$

Corollary 4 yields then

$$\begin{aligned} \frac{dF}{dX} &= -DD^{+}\left(X^{-1}\otimes X^{-1}\right) DD^{+} = -\tfrac{1}{4}(I+K)\left(X^{-1}\otimes X^{-1}\right)(I+K) = \\ &= -\tfrac{1}{4}(I+K)^{2}\left(X^{-1}\otimes X^{-1}\right) = -\tfrac{1}{2}(I+K)\left(X^{-1}\otimes X^{-1}\right). \end{aligned}$$

(3) $$\frac{dF}{dX} = \tfrac{1}{2}(I+K)(I\otimes X + X\otimes I) \text{ for } F = X^{2}.$$

We get

$$\begin{aligned} dF &= (dX)X + X\,dX, \\ d\,\mathrm{vec}F &= (I\otimes X + X\otimes I)\,d\,\mathrm{vec}X, \\ df &= D^{+}(I\otimes X + X\otimes I)D\,dx. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial f}{\partial x'} &= D^{+}(I\otimes X + X\otimes I)D, \\ \frac{dF}{dX} &= DD^{+}(I\otimes X + X\otimes I)DD^{+} = \\ &= \tfrac{1}{4}(I+K)(I\otimes X + X\otimes I)(I+K) = \tfrac{1}{4}(I+K)^{2}(I\otimes X + X\otimes I) = \\ &= \tfrac{1}{2}(I+K)(I\otimes X + X\otimes I). \end{aligned}$$